DERIVATION OF $\frac{\varepsilon_0}{\varepsilon_\infty} = \frac{\left\langle {\tau_D}^2 \right\rangle}{\left\langle \tau_D \right\rangle^2}$

NOTE: The subscripts 0 and ∞ refer to the low and high frequency limits respectively of a conductivity relaxation, not a dielectric relaxation that must necessarily occur over a higher frequency range in order that it not be hidden by the conductivity dielectric loss that is inversely proportional to frequency.

$$\varepsilon' = \frac{M'}{M'^2 + M''^2}$$

$$\lim_{\omega\to 0} M' = M_\infty \lim_{\omega\to 0} \int_0^\infty g\left(\tau_D\right) \frac{\omega^2 {\tau_D}^2}{1 + {\omega_D}^2 \tau^2} \, d\tau_D = M_\infty \omega^2 \left\langle {\tau_D}^2 \right\rangle$$

$$\lim_{\omega\to 0} M'' = M_\infty \lim_{\omega\to 0} \int_0^\infty g\left(\tau_D\right) \frac{\omega \tau_D}{1 + {\omega_D}^2 \tau^2} \, d\tau_D = M_\infty \omega \left\langle \tau_D \right\rangle$$

Thus

$$\varepsilon_0 \equiv \lim_{\omega\to 0} \varepsilon' = \frac{M_\infty \omega^2 \left\langle {\tau_D}^2 \right\rangle}{{M_\infty}^2 \left[\omega^4 \left\langle \tau_D \right\rangle^2 + \omega^2 \left\langle \tau_D \right\rangle^2 \right]} = \frac{1}{M_\infty} \frac{\omega^2 \left\langle {\tau_D}^2 \right\rangle}{\omega^2 \left\langle \tau_D \right\rangle^2} = \varepsilon_\infty \frac{\left\langle {\tau_D}^2 \right\rangle}{\left\langle \tau_D \right\rangle^2}$$

QED